Commentary on Buechele, Cooke, & Berezovsky (2024): Entropic Models of Scales and some Extensions

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ABSTRACT: I discuss Buechele, Cooke, and Berezovsky's entropy-based model of scale structure and compare it with a different entropy-based model from Milne et al. (2017). I also present an augmented version of the 2017 model to provide an additional entropy-based explanation for preferred scale structures. Our models have similarities and differences in terms of their constructions and constraints, and their results differ somewhat in meaning. Despite this, they are broadly comparable in terms of the "optimal" scales found. This suggests that entropy-based approaches can explain the origins of historical and contemporary scales, whilst also indicating interesting alternative scales that align with psychoacoustic and cognitive affordances.

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BUECHELE, Cooke, and Berezovsky (2024) present an intriguing entropy-based model to explain the origin of scales in Western music and their usage. Entropy is an appealing theoretical framework and has been broadly recognized as playing a role in music aesthetics, cognition, and perception (Loy, 2006; Milne & Herff, 2020; Pearce, 2018). For example, reducing the entropy of common signals (and conversely increasing entropy of uncommon signals) is the basis of Shannon's source coding theorem, which allows common signals to be losslessly compressed. Cognitive mechanisms performing such a role are likely important for the perception of musical structures (Milne, Dean, et al., 2023). Furthermore, entropy is a foundational part of Friston's free-energy principle (Friston, 2010). Of course, entropy is not a magic bullet because it can only be as important as the music-acoustical features over which it is calculated – these should be *perceptually relevant* features, and obtaining those is arguably the fundamental task (Milne & Herff, 2020).

Buechele, Cooke, and Berezovsky find that, depending on the temperature (the relative weighting of a scale's pitch classes' overall dissonance and the entropy of that scale's pitch class probabilities), different "scales" emerge. Here, I put quotes around "scales" because these are perhaps more accurately described as distributions of pitch-class [2] weights, as detailed subsequently.

With lower temperatures, higher pitch-class weights are found at just intonation-like diatonic pitch classes (bearing in mind that, in 5-limit just intonation [3], there are two possible choices for the supertonic degree of the major scale and, as the authors note, their model merges these into a broad peak). As the temperature is increased, a more meantone-like diatonic scale is highly weighted [4]. At a high enough temperature, all 12 chromatic pitches (equally spaced) emerge as strongly weighted. Using different critical bandwidths for the dissonance function, alternative high-temperature equal temperaments emerge, such as 19 and 31 – both of which are well-established microtonal chromatic scales that support meantone tunings of the diatonic scale and contain several pitch class intervals closely approximating just intonation (JI); including 7-limit JI intervals in the latter case.

A notable aspect of the Buechele, Cooke, and Berezovsky method is that the number of pitches in the scale are not constrained, and neither are the weights of the scale pitches. This means that the resulting distributions do not really correspond to what musicians or musicologists would typically classify as a scale: arguably, a scale is usually understood as a set of binary quantities, such that any given pitch (class) is either in the scale or not; for example, in music theory, scales are typically represented by binary indicator vectors, also known as characteristic functions (Amiot, 2016; Lewin, 1987; Milne et al., 2017); for example, the major scale would be represented as (1 0 1 0 1 1 0 1 0 1 0 1) where the ones and zeros show which degrees of the

12-tone chromatic scale are, respectively, included or not included in the scale. Instead, their method results in a measure of something that could be thought of as modelling each pitch class's "scalicness", or its "fit" to the scale or to a putative tonic or, as the authors suggest, a means to predict frequency-of-usage in Western tonal music. Furthermore, there is no strict procedure to count the number of distinct scale pitches implied by the distribution; for example, at what temperature can we say that the two supertonics in JI have merged into a single pitch class found in meantone? These are not intended as criticisms of their method because, as the authors demonstrate, the resulting hierarchies do seem to resemble the diachronic usage of tonal pitch classes in Western music as temperature is increased. Hence these non-binary distributions have explanatory utility.

In Milne et al. (2017), I followed an alternative entropy-based approach to model and optimize scale structure. It operated under considerably different conceptual and hence mathematical constraints. Interestingly, this method also produces meantone-like scales. The constraints in this model are that, prior to optimization, the number of pitch classes in each scale is fixed and each pitch class in that scale is equally weighted. Furthermore, the entropy that is optimized (minimized) is *spectral entropy* (Milne et al., 2017; Smit et al., 2019). Spectral entropy is the entropy of the Gaussian-smoothed and octave-reduced spectrum produced by harmonic complex tones with fundamentals at those scale pitches, and has proven useful in several contexts for predicting participants' responses to musical stimuli (Milne, Smit, et al., 2023; Smit et al., 2019; Smit et al., 2020, 2021).

Milne et al. (2017) is not referenced in the Buechele, Cooke, and Berezovsky article – this is quite understandable because it was little more than a sidenote in an article with a quite different focus: perfectly balanced rhythms and scales. Spectral entropy and its optimization under these constraints is detailed in Section 6.2 and the caption of Table 1 in Appendix B of Milne et al. (2017). There was insufficient space in that article to discuss the method in depth (and to have done so would have distracted from the main purpose of that paper), so the invitation to respond to the Buechele, Cooke, and Berezovsky article gives me an opportunity to provide some more information about this parallel entropy-based approach, provide a wider set of results from the model, discuss how the results parallel and contrast with theirs, run an extended version of the 2017 model using a new temperature-like function, and finally to suggest further research possibilities.

BRIEF OVERVIEW OF THE 2017 MODEL

A K-tone scale can be defined as a set of K pitch classes up to transposition; that is, any overall transposition of all pitch classes does not change the scale. For example, if C, D, E, G, A is classified as a "pentatonic" scale; the scale produced by shifting every pitch class by one semitone, by 0.2 semitones, or any other amount, is still "pentatonic".

A straightforward model of the overall perceptual "fit" of any set of pitch classes (such as the above pentatonic scale) is to consider the degree of spectral overlap produced by the tones in that scale – more overlap, better fit (Milne, 2013; Milne & Holland, 2016; Milne et al., 2015, 2016; Milne, Sethares, et al., 2011). To do this, we assume that every scale tone is a harmonic complex tone (i.e., it has harmonic partials – such tones are predominant in Western music, largely because they produce a strong and unambiguously perceived pitch; unlike tones with non-harmonic partials whose pitch is typically ambiguous or absent). The resulting spectrum is smoothed, in the log-frequency domain, with a gaussian kernel to model inaccuracies of pitch perception and "folded" into a single octave to model octave equivalence. The entropy of this distribution is called *spectral entropy* and, as cited earlier, it has been shown to be an effective predictor of affective responses to musical harmony.

For a K-tone scale, the optimization process generates $K - 1$ random pitches within one octave (one pitch is always 0, so including this makes a total of K pitch classes). From that starting point, MATLAB's `fmincon` function finds a local minimum for spectral entropy by adjusting the pitch values under the constraint that no two pitches can come closer than 30 cents. (This constraint is required because if pitch classes are allowed to become too close, the scale with minimal spectral will always be a scale where all K tones have identical pitch classes; given the smoothing width of 6 cents, setting the constraint lower than 30 sometimes results in the optimal scale hitting that constraint as they try to approach 0.) Using the "hit-and-run" optimization method, once the local minimum is found, another random set of pitches is generated, and the process continues but now under the constraint that any previously found minimum is avoided by assigning it with an infinite cost. Five hundred such searches for local minima are done, which is sufficient to find the global minimum. This process is then repeated to find the second-best scale (with all previous local minima avoided), the third best scale, and so on until the top-five K-tone scales are found.

The above description of the calculations of spectral entropy and the optimization is brief and leaves out the mathematical details. Those interested in specifics of the implementation can download the code from https://osf.io/3etnb/ (this includes the original code in the online supplementary for Milne et al. (2017) available at http://www.dynamictonality.com/perfect_balance_files/).

RESULTS OF THE 2017 MODEL

In Milne et al. (2017), the model was primarily used to find the "best" perfectly balanced 5- to 12-tone scales (*perfect balance* is rather interesting constraint on scale and rhythm structure, which was the focus of that article – in a perfectly balanced scale, the distribution of pitch classes, represented as points on a circle, has maximal circular variance [5]) and so included an additional (but optional) nonlinear constraint that the resulting scale was, indeed, perfectly balanced. As reported in Milne et al. (2017, p. 123 & 132), without that constraint, the 7-tone scale with minimal spectral entropy closely corresponds to a meantone diatonic scale with perfect fifths averaging 697.60 cents.

Here, I report the top-five (lowest spectral entropy) scales with 3- to 13-tones. The only constraints are that all pitch classes are equally weighted, and each optimization is for a specific number of such pitch classes. These are calculated with the same code and parameter values used in Milne et al. (2017): the number of harmonics in each tone is 12, the amplitude of each tone's nth harmonic is $n^{-0.6}$, and the Gaussian smoothing kernel has a standard deviation of 6 cents. These parameter values correspond to those optimized to data obtained from earlier music perception experiments (Milne & Holland, 2016; Milne et al., 2015, 2016). In Table 1, scale pitch classes are shown in cents, with the entropy of each scale's spectrum in the final column.

Table 1. Scales with 3 to 12 tones: for each number of tones, the five scales with the lowest spectral entropy.

Scale pitch classes (cents, arranged so smallest interval comes first)							*Spectral entropy*
3-tone scales							
0.00	203.68	701.84					–1.7649
0.00	315.66	813.66					–1.7273
0.00	182.51	498.09					–1.6780
0.00	315.67	702.01					–1.6652
0.00	230.44	932.58					–1.6643
4-tone scales							
0.00	182.02	497.81	995.65				–1.5502
0.00	201.01	499.24	700.64				–1.5469
0.00	111.66	498.02	813.64				–1.5224
0.00	182.45	498.04	884.38				–1.5211
0.00	111.90	315.63	813.72				–1.5112
5-tone scales							
0.00	192.65	501.98	696.01	1001.95			–1.4021
0.00	111.55	315.89	497.94	813.70			–1.3997
0.00	113.03	315.46	813.99	1016.56			–1.3672
0.00	111.73	498.14	813.62	996.38			–1.3622
0.00	113.15	498.61	612.27	814.63			–1.3611
6-tone scales							
0.00	111.47	311.13	501.21	811.73	1007.32		–1.2805
0.00	112.85	316.06	498.57	701.16	814.30		–1.2704
0.00	113.52	315.38	498.73	613.59	814.45		–1.2572
0.00	118.18	313.50	618.47	815.43	1010.43		–1.2567
0.00	71.24	386.62	498.05	701.78	884.83		–1.2469
7-tone scales							
0.00	113.18	312.02	502.72	698.78	812.85	1009.82	–1.1837
0.00	81.30	196.71	392.96	580.93	699.57	894.88	–1.1542
0.00	71.89	386.47	498.37	701.51	884.93	1087.71	–1.1491
0.00	78.20	199.02	390.69	699.97	893.62	1089.43	–1.1475

Scale pitch classes (cents, arranged so smallest interval comes first)	*Spectral entropy*
0.00 75.43 195.25 389.09 499.86 698.84 889.74	–1.1461
8-tone scales	
0.00 82.47 197.96 390.85 584.02 699.51 895.22 1087.01	–1.0916
0.00 75.77 197.06 388.26 500.00 699.02 889.68 1086.77	–1.0714
0.00 77.71 198.47 390.58 699.84 777.98 893.29 1089.96	–1.0668
0.00 73.19 190.89 387.40 501.08 697.03 888.06 1001.76	–1.0663
0.00 79.07 386.64 498.97 581.56 700.57 886.86 1084.57	–1.0657
9-tone scales	
0.00 80.14 195.30 388.64 500.35 581.77 698.35 891.18 1084.70	–1.0161
0.00 74.95 196.19 387.99 500.27 698.81 774.88 889.38 1087.21	–0.9976
0.00 74.08 191.59 304.40 387.89 502.67 694.99 888.82 1004.30	–0.9960
0.00 73.91 192.79 386.82 501.45 696.94 888.52 1002.47 1085.11	–0.9926
0.00 73.17 190.92 387.30 501.27 697.02 773.66 888.07 1001.53	–0.9915
10-tone scales	
0.00 78.42 192.47 387.21 501.59 580.46 696.83 889.93 1002.32 1083.45	–0.9470
0.00 74.62 193.20 304.74 387.19 502.82 695.60 889.11 1004.78 1085.30	–0.9337
0.00 73.58 192.46 386.85 501.48 697.04 773.50 888.41 1002.03 1085.84	–0.9310
0.00 74.81 195.89 274.00 387.76 499.80 698.50 774.37 889.44 1086.76	–0.9305
0.00 76.77 193.51 308.59 389.71 504.21 697.11 808.87 890.76 1007.38	–0.9270
11-tone scales	
0.00 77.56 192.64 387.26 501.83 580.74 697.03 775.41 889.88 1002.12 1084.50	–0.8888
0.00 74.30 192.95 304.17 387.22 502.87 695.50 774.52 889.02 1004.34 1085.95	–0.8744
0.00 77.22 194.71 308.82 388.81 504.17 697.34 809.50 890.82 1007.72 1086.53	–0.8695
0.00 73.67 192.11 273.79 386.78 500.84 696.83 773.28 888.53 1000.95 1085.64	–0.8682
0.00 74.32 186.44 269.96 384.69 499.18 577.98 771.17 886.07 997.93 1082.08	–0.8637
12-tone scales	
0.00 77.70 192.48 273.19 387.05 501.27 581.47 696.79 774.52 889.85 1001.35 1084.36	–0.8336
0.00 41.07 119.14 235.10 314.19 428.50 542.29 622.45 738.75 815.47 931.56 1125.73	–0.8170
0.00 34.88 117.55 232.74 311.51 425.48 620.18 734.64 813.57 930.42 1009.12 1122.93	–0.8148
0.00 30.00 113.13 228.53 421.39 500.05 615.15 729.92 811.67 926.29 1000.25 1118.75	–0.8139
0.00 32.36 114.12 230.69 310.76 423.38 500.13 617.91 731.75 812.21 927.77 1120.68	–0.8125

In the following text, I use these abbreviations for intervals: P = "perfect", M = "major", m = "minor", A = "augmented", d = "diminished". The optimal 3-tone scale is very similar to a 3-tone *well-formed* (WF) [6] scale making up the primary degrees (tonic, subdominant, dominant) of the diatonic scale (this scale is sometimes called the *tetractys* (de Jong & Noll, 2011)) and is here generated by perfect fifths very close to JI (the average P5 size is 701.84 cents; a JI 3/2 P5 is 701.96).

The optimal 4-tone scale is not well formed and is not generated by a continuous chain of fifths.

The optimal 5-tone scale is very similar to the 5-tone WF scale generated by fifths, which has a *signature* of "2L 3s" (i.e., it is the anhemitonic pentatonic scale, which has 2 large steps and 3 small) [7]. The average size of the fifth in this optimal 5-tone scale is 697.68, hence is a slightly flat (from JI) meantone perfect fifth – this slight flattening means that the major third resulting from stacking four perfect fifths and the minor third resulting from stacking three perfect fifths more closely approximate their JI referents.

The optimal 6-tone scale is very similar to a scale generated by a chain of slightly flat perfect fifths (average size = 687.71 cents) but, due to it comprising 6 pitch classes, is not WF.

The optimal 7-tone scale is very similar to a 7-tone scale generated by perfect fifths with a mean size of 697.60 cents (ranging from 692.90 to 699.67). If we label the pitches as in a white-note diatonic scale, we see that the fifths F–C, C–G, G–D are larger than the fifths D–A, A–E, and E–B. This is because slightly flattening the latter three fifths makes the major thirds F–A, C–E, and G–B closer to the JI tuning of 5/4. In a sense, this scale tuning is a slightly irregular "well-temperament" for a diatonic scale – analogous to the well-temperaments historically applied to the full 12-tone chromatic scale (e.g., Werckmeister and Kirnberger (Barbour, 1951)).

The optimal 8-tone scale is not similar to a scale generated by perfect fifths (it is a major scale with an additional m6/A5). The optimal 9-tone scale is also not similar to a scale generated by single chain of fifths (it is a major scale with an additional m6/A5 and m2/A1). The optimal 10-tone scale adds an A4/d5, hence is similar to two chains-of-fifths B♭–F–C–G–D–A–E and B–F♯–C♯. The optimal 11-tone scale adds a minor third; in so doing, the scale becomes similar to a well-formed scale 1L 10s, which generated by equal-sized semitones E–F–F♯–G–G♯–A–A♯–B–C–C♯–D; it is also, of course, rich in P5s.

The optimal 12-tone scale is similar to the familiar chromatic scale generated by 11 meantone fifths (average size 697.31 cents), where the 7 (large) diatonic semitones and 5 (small) chromatic semitones (hence this well-formed scale has the signature 7L 5s) are distributed as evenly as possible; for example, if the pitch class of 192.48 (from the optimal 12-tone tuning shown in the above table) is C, the resulting scale is: C–C♯–D–E♭–E–F–F♯–G–A♭–A–B♭–B.

This convergence towards meantone-like tunings of pentatonic, diatonic, and chromatic has some parallel to the results found by Buechele, Cooke, and Berezovsky. However, one might wonder if this 2017 model can inform about preferences found in Western music and probably some non-Western traditions for the pentatonic, diatonic, and 12-tone chromatic scales over, for example, 4–6-tone scales and 8–11-tone scales? In the Buechele, Cooke, and Berezovsky model, as the temperature increases, their Figure 3 shows three prominent peaks (the tetractys), then perhaps two more peaks emerge forming a non-WF pentatonic scale (C–E–F–G–A♭), then two more peaks fill in the gaps to produce the meantone diatonic scale; finally the 12-tone equal scale emerges. However, there is not a natural threshold to say whether a pitch class has emerged sufficiently to assert it is actually a scale member (as before, this is not a criticism of their model, which is arguably more a model of pitch class usage than of scales per se).

For my 2017 model, one might think, naively, that we could simply compare the entropies between scales with different numbers of tones. However, as additional scale-tones are added, entropy will inevitably rise and there is no natural normalization process (that I am aware of) that can discount for the increased numbers of tones given the restricted support of the distribution (one circular octave) [8]. But there is an additional quantification of entropy that can augment the 2017 model. This may provide some insight into why the 5-, 7- and 12-tone WF scales may be special. I call this the 2024 scalic entropy model.

BRIEF OVERVIEW OF THE 2024 SCALIC ENTROPY MODEL

This augmented version of the 2017 model considers not just the entropy of a scale's spectrum, but also the entropy of its step sizes, both of which are optimized (made as low as possible). Note that the optimal spectral entropy 3-tone, 5-tone, 7-tone, 11-tone, and 12-tone scale all approximate well-formed scales. As detailed earlier, WF scales have no more than two step sizes: the 3-tone scale has two large steps (P4s) and one small step (M2); the 5-tone (anhemitonic pentatonic) scale has 2 large steps (m3s) and 3 small steps (M2s); the 7-tone diatonic scale has 5 large steps (M2s) and 2 small steps (m2s); the 12-tone chromatic scale has 5 large steps (m2s, also called diatonic semitones) and 7 small steps (augmented unisons, also called chromatic semitones).

A scale structured from one or two different step sizes clearly has a simpler, more guessable, structure than scales with more than two step sizes. A simple way to quantify this complexity is to measure the entropy of all K step sizes in a K-tone scale, while also applying Gaussian smoothing to model perceptual inaccuracies. A related model of interonset interval entropy has proved to be a useful predictor for modelling the perception of complex rhythms (Milne & Herff, 2020) but not for their performance (Milne, Dean, et al., 2023). This scale-step entropy provides a second cost function that can be minimized and, similar to how the "temperature" in the Buechele, Cooke, and Berezovsky model relatively weights the dissonance cost with the pitch-class distribution entropy, so we can introduce a parameter α to adjust the relative weights of the spectral energy and scale-step entropy costs (a notable difference is that scale-step entropy is minimized in my model, pitch-class distribution entropy is maximized in their model). This is done by simply adding ($\alpha \times \text{scale-step entropy}$) to spectral entropy; hence, when $\alpha = 0$, this model corresponds to the 2017 model. Indeed, the 2017 model can be thought of as related to Buechele, Cooke, and Berezovsky's low-temperature model where roughness has been substituted by spectral entropy and the additional constraints supplied.

As α increases, we find that the well-formed-like scales (e.g., the 2017 model's optimal 7-tone scale) more closely approach their ideal WF forms (e.g., the fifths become more similar in size). This is unsurprising because WF scales, by definition, have no more than two step sizes and so have lower scale-step entropy than scales with more step sizes; because Gaussian smoothing of the step interval sizes is applied, as they approach the WF ideal, their scale-step entropy smoothly decreases. As α is increased further, the scale will ultimately shift to a perfectly equal scale. So, for 5-tones, the scale switches from the WF pentatonic, then to 5-tone equal. It is interesting to observe the order of these switches – from irregular to WF to equal – by alpha and scale size; this is shown in Table 2.

Table 2. This table shows the α value (to a resolution of 0.025) at which each scale with a given number of pitch classes "snaps" to an essentially well-formed (WF) or equally tuned (E) structure. In some cases, the scale forms a non-WF subset of an equal tuning (e.g., Sub12 is a non-WF subset of 12-TET). The approximations of the scale's intervals to JI ratios are represented by their temperament family [9] which is indicated with the following symbols: * = Meantone, † = Ripple, ‡ = Valentine, § = Porcupine, ¶ = Hemififths.

	Scale size											
α	3	4	5	6	7	8	9	10	11	12	13	14
0	WF		WF*		WF*					WF*		
0.025												
0.05												
0.075										E*†		
0.1												
0.125									WF†			
0.15							Sub12					WF‡
0.175						WF§					Sub24	
0.2												
0.225												
0.25		WF¶						WF†				
0.275												
0.3											WF‡	
0.325				E								
0.35												
0.375												
0.4	E											
0.425			E									
0.45												
0.475		E										
0.5												

As α increases much beyond 0.3, the scales are evidently less influenced by spectral considerations; for example, the rather dissonant (but lower scale-step entropy) 5-TET scale becomes preferred over the more consonant (but higher scale-step entropy) 5-tone WF pentatonic scale. It is interesting to note that the 3-, 5-, 7-, and 12-tone scales are all essentially WF even with no influence of step entropy (i.e., when $\alpha = 0$). As α is increased for these scales, they become closer to their well-formed ideal. For example, when $\alpha = 0.3$, the six perfect fifths in the diatonic scale range from 696.65 to 698.13, with a mean of 697.30 (when $\alpha = 0$, they range from 692.90 to 699.67 with a mean of 697.60); when $\alpha = 0.075$, the step sizes of the equal 12-tone scale are all exactly 100.00 cents (and all perfect fifths are 700.00 cents).

This is because the 3-, 5-, 7-, and 12-tone WF scales already have low step entropies and, of these, only the 12-tone WF scale is close to an even lower scale-step entropy 12-tone equal scale that is also low in spectral entropy; indeed, even at $\alpha = 0.5$, the WF 7-tone diatonic scale is preferred over 7-TET, which suggests that the diatonic scale is an unusually robust choice given that it is optimal over such a wide range of α (relative weightings of spectral and scale-step entropy). This matters because listeners/musicians/composers may differentially weight these two entropies so this scale should be optimal for a broad range of individuals for whom these two entropies are pertinent.

There are also some interesting WF microtonal scales appearing at relatively low values of α. The 8-tone WF scale (appearing when $\alpha = 0.175$) is the Porcupine scale [9], which is fabulously microtonal sounding (it sounds very unlike familiar Western tunings) whilst also comprising several good approximations of JI intervals; it is a temperament I have made use of in previous compositions (e.g., https://soundcloud.com/andrew-j-milne/yak-butter-1 and https://soundcloud.com/andrew-j-milne/once-seen-initial-sketch-2024). The 13- and 14-tone Valentine scales [9] comprise a long chain of chromatic semitones (~25/24), 4 of which are close to 6/5, 5 of which are close to 5/4, and 9 of which are close to 3/2; this tuning is closely related to Wendy Carlos' Alpha tuning as used on the title track of her Beauty and the Beast album (Carlos, 1986; Milano, 1986). It is also interesting to note that these attractively simple microtonal WF scales do not arise at all in the top-5 lists produced by the 2017 model (i.e., where α is implicitly 0).

CONCLUSION

In summary, entropy seems a valuable tool for explaining the origins of historical and contemporary scales, as well as for pointing towards possible future alternative scales that align with psychoacoustic and cognitive affordances. There are several useful music-acoustical features over which entropy can be calculated, and different types of constraints that can be applied over its optimization. These suggest some future research avenues. For scales constrained to have equally weighted pitch classes (as in my 2017 and 2024 models), it would be interesting to explore the entropies of different music-acoustical features; for example, instead of scale-step entropy, we could use the entropy of sequences of intervals between three or more adjacent pitch classes (trichords, tetrachords, pentachords, etc.), which favour well-formed arrangements of two step sizes over non-well-formed arrangements; or, alternatively, the entropy of all intervals found in the scale could be calculated. For the Buechele, Cooke, and Berezovsky model, it could be interesting to try spectral entropy instead of roughness; one possible reason this may be useful is because roughness is designed for simultaneously sounded tones whereas spectral entropy is also applicable to melodically presented (non-simultaneous) intervals, and there is an appealing symmetry to all aspects of the model being entropy-based.

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NOTES

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[2] The term *pitch class*, here, indicates only octave equivalence, it does not imply any specific division, e.g. 12-fold, of the octave – equal or otherwise.

[3] A *just intonation* (JI) tuning is one where every interval is a rational frequency ratio. In a p-limit JI, the numerator and denominator of these rational frequency ratios are factorizable by primes no higher than p.

[4] *Meantone* refers to any tuning where the two supertonics – 9/8 and 10/9 in JI, which differ by the syntonic comma 81/80 – are represented by the same pitch (i.e., the syntonic comma is tempered out). Meantone tunings typically share the tuning "errors", relative to just intonation (JI), between the perfect fifths, major thirds, and minor thirds (and their octave inversions). Assuming a temperament generated by perfect fifths and an octave of 2/1, this implies the fifths must be tuned in the open interval (694.79, 701.96) cents; that is, from 1/3-comma meantone, which has JI minor thirds (6/5), to Pythagorean, which has JI perfect fifths (3/2) (Milne et al., 2006). Perfect fifths around 696–699 cents share the errors fairly evenly between fifths and major and minor thirds and are the typical range of sizes actually used (Milne, Sethares, et al., 2011).

[5] In a *perfectly balanced* scale (or rhythm), the distribution of pitch classes (or onsets), represented as points on a circle, has maximal circular variance. In 12-TET, perfectly balanced scales include perfectly even scales/chords such as the whole tone and diminished seventh chord, and modes of limited transposition such as the hexatonic and octatonic, but also interesting irregular scales such as the double harmonic minor (e.g., C–D♭–E–F–G–G♯–A). Perfectly balanced scales and rhythms take particularly interesting forms when they are subsets of a regular subdivision of the octave (or rhythmic period) where the number of such subdivisions

is a product of two or more distinct primes, hence they provide a useful strategy for exploring unfamiliar tunings and meters (Milne et al., 2017).

[6] A *well-formed* scale comprises no more than two step sizes arranged as evenly as possible (Carey & Clampitt, 1989; Milne, Carlé, et al., 2011; Milne & Dean, 2016; Wilson, 1975). An unbroken chain of generating intervals of the same size (subsequently reduced by octaves) will produce well-formed scales for specific sizes of chain; for example, given a just-intonation fifth (3/2) generator, the following infinite sequence of numbers of tones will produce a WF scale: 3, 5, 7, 12, 17, 29, 34, …. Differently sized generators have different such sequences.

[7] Well-formed scales can be characterized by their *signature*, which is their numbers of large (L) and small (s) steps; e.g., the familiar anhemitonic pentatonic scale is 2L 3s, the diatonic scale is 5L 2s (Milne, Carlé, et al., 2011; Wilson, 1975). Several other signatures exist in unfamiliar microtonal tunings.

[8] A familiar method for normalizing discrete entropy is to divide it by $\log N$, where N is the size of the distribution's support (for spectral entropy, if pitch classes are discretized to the nearest cent, $N = 1200$). This is because $\log N$ is the maximum entropy for such a support (it is the entropy when the probability distribution is flat). However, the aim is to account for the number of tones in the scale, not the size of the support. To normalize for this, requires calculating the maximum spectral entropy for 1 tone, for 2 tones, for 3 tones, etc. Given that each tone is itself a mixture distribution of Gaussians (one for each harmonic), and that combinations of such tones will always overlap to some extent (due octave equivalence), I am not aware of any analytical method to calculate these maximum entropies.

[9] A temperament family represents a mapping from just intonation to a lower-dimensional tuning such as the two-dimensional tunings that generate WF scales. Because of the dimensionality reduction, the resulting tuning requires some or all of the just intonation tunings to be tempered (mistuned by a small amount) (Milne et al., 2008). See Erlich (2006) and the Xenharmonic Wiki (https://en.xen.wiki/) for precise specifications of each these temperament families.

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